Momentum Maps and Measure-valued Solutions (Peakons, Filaments and Sheets) for the EPDiff Equation

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To Alan Weinstein on the Occasion of his 60th Birthday

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Abstract

We study the dynamics of measure-valued solutions of what we call the EPDiff equations, standing for the *Euler-Poincaré equations associated with the diffeomorphism group (of* $\mathbb{R}^n$ *or an* n*-dimensional manifold* M*)*. Our main focus will be on the case of quadratic Lagrangians; that is, on geodesic motion on the diffeomorphism group with respect to the right invariant Sobolev H^1 metric. The corresponding Euler-Poincaré (EP) equations are the EPDiff equations, which coincide with the averaged template matching equations (ATME) from computer vision and agree with the Camassa-Holm (CH) equations in one dimension. The corresponding equations for the volume preserving diffeomorphism group are the well-known LAE (Lagrangian averaged Euler) equations for incompressible fluids.

We first show that the EPDiff equations are generated by a smooth vector field on the diffeomorphism group for sufficiently smooth solutions. This is analogous to known results for incompressible fluids—both the Euler equations and the LAE equations—and it shows that for sufficiently smooth solutions, the equations are well-posed for short time. In fact, numerical evidence suggests

that, as time progresses, these smooth solutions break up into singular solutions which, at least in one dimension, exhibit soliton behavior.

With regard to these non-smooth solutions, we study measure-valued solutions that generalize to higher dimensions the peakon solutions of the (CH) equation in one dimension. One of the main purposes of this paper is to show that many of the properties of these measure-valued solutions may be understood through the fact that their solution ansatz is a momentum map. Some additional geometry is also pointed out, for example, that this momentum map is one leg of a natural dual pair.

Contents

1 Introduction

This paper is concerned with solutions of the EPDiff equations; that is, with the Euler-Poincaré equations associated with the diffeomorphism group in n-dimensions. In particular, we are concerned with singular solutions that generalize the peakon solutions of the Camassa-Holm (CH) equation in one dimension. The CH equation (see Camassa and Holm [1993]) for the dynamics of shallow water in a certain asymptotic regime, is

$$u_t + 3uu_x = \alpha^2 \left(u_{xxt} + 2u_x u_{xx} + u u_{xxx} \right), \tag{1.1}$$

where $u(x,t)$ is the fluid velocity, subscripts denote partial derivatives in position x and time t, and α^2 is a positive constant. (See Dullin, Gottwald and Holm [2001, 2003] for recent discussions of the derivation and asymptotic validity of the CH equation for shallow water waves, at one order beyond the Korteweg-de Vries equation.) Equivalently, in Hamiltonian form, the CH equation reads

$$m_t + u m_x + 2u_x m = 0 \tag{1.2}$$

where $m = u - \alpha^2 u_{xx}$ and α^2 is a positive constant. As Camassa and Holm [1993] show, in Hamiltonian form the CH equation is the Lie-Poisson equation associated with the Lie algebra of one dimensional vector fields and with the Hamiltonian

$$h(m) = \frac{1}{2} \int u\, m\, dx. \tag{1.3}$$

The CH equation may be equivalently expressed in Euler-Poincaré form by using the Lagrangian associated with the H^1 metric for the fluid velocity. That is, the Lagrangian as a function of the fluid velocity is given by the quadratic form,

$$l(u) = \frac{1}{2} \int (u^2 + \alpha^2 u_x^2)\, dx. \tag{1.4}$$

It follows from Euler-Poincaré theory (see Marsden and Ratiu [1999] and Holm, Marsden and Ratiu [1998a]) that the one parameter curve of diffeomorphisms $\eta(x_0, t)$ depending on parameter t and defined implicitly by

$$\frac{\partial}{\partial t}\eta(x_0, t) = u(\eta(x_0, t), t)$$

is a geodesic in the group of diffeomorphisms of $\mathbb{R}$ (or, with periodic boundary conditions, of the circle S^1) equipped with the right invariant metric equal to the H^1 metric at the identity.

A remarkable analytical property of the CH equation, conjectured by keeping track of derivative losses in Holm, Marsden and Ratiu [1998a] and proved in Shkoller [1998] is that the geodesic equations literally define a smooth vector field in the Sobolev H^s topology. That is, in the material representation, the equations have no derivative loss. This property is analogous to the corresponding results for the Euler equations for ideal incompressible fluid flow (discovered by Ebin and Marsden [1970]) and the Lagrangian averaged Euler equations (again conjectured by Holm, Marsden and Ratiu [1998a] and proved by Shkoller [1998]).

As we will explain in §3, a similar statement holds for the n-dimensional EPDiff equation if we use the H^1 metric. This is all the more remarkable because smoothness of the geodesic flow is *not true* for the L^2 metric, at least not without assuming incompressibility. Smoothness of *volume-preserving* geodesic flow with respect to the L^2 metric does hold for the incompressible flow of an ideal Euler fluid, a result proved in Ebin and Marsden [1970].

Before proceeding with a discussion of the general case of the n-dimensional EPDiff equations, we shall quickly review, mostly to establish notation, a few facts about the Euler-Poincaré and Lie-Poisson equations, whose basic theory is explained, for example, in Marsden and Ratiu [1999].

Review of Euler-Poincaré and Lie-Poisson Equations. Let G be a Lie group and $\mathfrak{g}$ its associated Lie algebra (identified with the tangent space to G at the identity element), with Lie bracket denoted by $[\xi, \eta]$ for $\xi, \eta \in \mathfrak{g}$. Let $\ell : \mathfrak{g} \to \mathbb{R}$ be a given Lagrangian. Let $L : TG \to \mathbb{R}$ be the right invariant Lagrangian on G obtained by translating ℓ from the identity element to other points of G via the right action of G on TG. A basic result of Euler-Poincaré theory is that the Euler–Lagrange equations for L on G are equivalent to the (right) Euler-Poincaré equations for ℓ on $\mathfrak{g}$, namely to

$$\frac{d}{dt}\frac{\delta \ell}{\partial \xi} = -\operatorname{ad}^*_\xi \frac{\delta \ell}{\partial \xi}\,. \tag{1.5}$$

Here, $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ is the adjoint operator, the linear map given by the Lie bracket $\eta \mapsto [\xi, \eta]$. Also, $\mathrm{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ is its dual; that is, $\left\langle \mathrm{ad}^*_\xi(\mu), \eta \right\rangle = \langle \mu, [\xi, \eta] \rangle$, where $\langle \, , \rangle$ is the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Also, $\delta\ell/\delta\xi$ denotes the functional derivative of ℓ with respect to $\xi \in \mathfrak{g}$. For *left* invariant systems, we change the sign in (1.5). The Euler-Poincaré equations can be written in the variational form

$$\delta \int \ell \, dt = 0 \, , \tag{1.6}$$

for all variations of the form $\delta\xi = \dot{\eta} - [\xi, \eta]$ for some curve η in $\mathfrak{g}$ that vanishes at the endpoints.

If the reduced Legendre transformation $\xi \mapsto \mu = \delta\ell/\delta\xi$ is invertible, then the Euler-Poincaré equations are equivalent to the (right) Lie-Poisson equations:

$$\dot{\mu} = -\,\mathrm{ad}^*_{\delta h/\delta\mu} \, \mu \, , \tag{1.7}$$

where the reduced Hamiltonian is given by,

$$h(\mu) = \langle \mu, \xi \rangle - \ell(\xi).$$

These equations are equivalent (via Lie-Poisson reduction and reconstruction) to Hamilton's equations on T^*G relative to the Hamiltonian $H : T^*G \to \mathbb{R}$, obtained by right translating h from the identity element to other points via the right action of G on T^*G. The Lie-Poisson equations may be written in the Poisson bracket form

$$\dot{F} = \{F, h\} \, , \tag{1.8}$$

where $F : \mathfrak{g}^* \to \mathbb{R}$ is an arbitrary smooth function and the bracket is the (right) Lie-Poisson bracket given by

$$\{F, G\}(\mu) = \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right] \right\rangle . \tag{1.9}$$

In the important case when ℓ is quadratic, the Lagrangian L is the quadratic form associated to a right invariant Riemannian metric on G. In this case, the Euler–Lagrange equations for L on G describe geodesic motion relative to this metric and these geodesics are then equivalently described by either the Euler-Poincaré, or the Lie-Poisson equations.

Outline of the paper. The main results of the present paper are as follows:

1. In §2 we review some basic facts about the EPDiff equations, especially the *singular solution ansatz* (2.8) of Holm and Staley [2003] that introduces a class of singular solutions that generalize the peakon solutions of the CH equation to higher spatial dimensions.

2. In §3 we give a plausibility argument that the EPDiff equations possess an interesting *smoothness property*; namely, they define a smooth vector field (that is, they define ODE's with no derivative loss) in the Lagrangian representation.

This means, in particular, that the EPDiff equations are locally well posed for sufficiently smooth initial data. Because of the development of singularities in finite time, which the numerics suggests, the smooth solutions may not exist globally in time. This smoothness property is similar to the corresponding smoothness property of the Euler equations for ideal incompressible fluid mechanics shown in Ebin and Marsden [1970].

3. In §4, we show that the singular solution ansatz (2.8) defines an equivariant momentum map. We do this in a natural way by identifying the singular solutions with certain curves in the space of embeddings $\mathrm{Emb}(S, \mathbb{R}^n)$ of a generally lower dimensional manifold S into the underlying manifold $\mathbb{R}^n$ (or an n-manifold M) and letting the diffeomorphism group act on this space. The right action of $\mathrm{Diff}(S)$ corresponds to the right invariance of the EPDiff equations, while the left action of $\mathrm{Diff}(\mathbb{R}^n)$ gives the desired solution ansatz.

4. In §5 we explore the geometry of the singular solution momentum map, in parallel with the corresponding work on singular solutions (vortices, filaments, etc.) for the Euler equations of an ideal fluid in Marsden and Weinstein [1983].

5. Finally, in §6, we discuss some of the remaining challenges and speculate on some of the many possible future directions for this work.

Historical Note. This paper is dedicated to our friend and collaborator Alan Weinstein and, for us, this work parallels some of our earlier collaborations with him. Alan's basic works on reduction, Poisson geometry, semidirect product theory, and stability in mechanics—just to name a few areas—have been, and remain incredibly influential and important to the field of geometric mechanics. See, for instance, Marsden and Weinstein [1974]; Weinstein [1983b]; Marsden, Ratiu and Weinstein [1984]; Weinstein [1984]; Holm, Marsden, Ratiu and Weinstein [1985]

Mechanics on Lie groups was pioneered by Arnold [1966], a reference that is a key foundation for the subject and in particular for the present paper. However, this theory was in a relatively primitive state, even by 1980, and it has benefited greatly from Alan's insights. In fact, the clear distinction between the Euler-Poincaré and Lie-Poisson equations, the former with a variational structure and the latter with its Poisson structure, which took until the 1980's to crystalize, was greatly aided by Alan's work.

Alan has made key contributions to many fundamental concepts in geometrical mechanics, such as Lagrangian submanifolds and related structures (Weinstein [1971, 1977]), symplectic reduction (Marsden and Weinstein [1974]), normal modes and periodic orbits (Weinstein [1973, 1978]), Poisson manifolds (Weinstein [1983b]), geometric phases (Weinstein [1990]), Dirac structures (Courant and Weinstein [1988]) groupoids and Lagrangian reduction (Weinstein [1996]) and the plethora of related "oid" structures he has been working on during the last decade (just look over the over 200 papers on MathSciNet he has written!) that will surely play an important role in the next generation of people working in the area of geometric mechanics.

Of Alan's papers, the one that is most directly relevant to the topics discussed in the present paper is Marsden and Weinstein [1983]. Alan himself is is still developing this area, too, as in Weinstein [2002].

2 The EPDiff Equation

In this section we review the EPDiff equation; that is, the Euler-Poincaré (EP) equation associated with the diffeomorphism group in n-dimensions. This equation coincides with a limiting case of the CH equation for shallow water waves in one and two dimensions. It also coincides with the ATME equation (the averaged template matching equation) in two dimensions. The latter equation arises in computer vision; see, for instance, Mumford [1998]; Hirani, Marsden and Arvo [2001] or Miller, Trouve and Younes [2002] for a description and further references. We have chosen to call this by a generic name, the ***EPDiff equation***, because it has these various interpretations in different applications. Of course these different interpretations also provide opportunities: for example, to see to what extent the singular solutions found in the EPDiff equations are applicable, either for shallow water wave interactions, or for computer vision applications. A recent combination of these ideas in which image processing imitates soliton interactions appears in Holm, Trouvé and Younes [2003].

Statement of the EPDiff Equations. Treating analytical issues formally at this point, let $\mathfrak{X}$ denote the Lie algebra of vector fields on an n-dimensional manifold M (such as $\mathbb{R}^n$). The vector fields comprise the algebra associated with the diffeomorphism group of M, but the usual Jacobi-Lie bracket is the negative of the (standard) Lie algebra bracket. (See Marsden and Ratiu [1999] for a discussion.)

Let $\ell : \mathfrak{X} \to \mathbb{R}$ be a given Lagrangian and let $\mathfrak{M}$ denote the space of one-form densities on M, that is, the momentum densities. The corresponding momentum density of the fluid is defined as

$$m = \frac{\delta \ell}{\delta u} \in \mathfrak{M} \,,$$

which is the functional derivative of the Lagrangian ℓ with respect to the fluid velocity $u \in \mathfrak{X}$. If u is the basic dynamical variable, the EPDiff equations are simply the Euler-Poincaré equations associated with this Lagrangian. Equivalently, if m is the basic dynamical variable, a Legendre transformation allows one to identify the EPDiff equations as the Lie-Poisson equations associated with the resulting Hamiltonian. For the case of $\mathbb{R}^n$, we will use vector notation for the momentum density $\mathbf{m}(\mathbf{x}, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ (a bold $\mathbf{m}$ instead of a lightface m). The EPDiff equations are as follows (see Holm, Marsden and Ratiu [1998a,b, 2002] for additional background and for techniques for computing the Euler-Poincaré equations for field theories),

$$\frac{\partial}{\partial t}\mathbf{m} + \underbrace{\mathbf{u} \cdot \nabla \mathbf{m}}_{\text{convection}} + \underbrace{\nabla \mathbf{u}^T \cdot \mathbf{m}}_{\text{stretching}} + \underbrace{\mathbf{m}\,(\operatorname{div} \mathbf{u})}_{\text{expansion}} = 0 \,. \tag{2.1}$$

In coordinates x^i, $i = 1, 2, \ldots, n$, using the summation convention, and writing $\mathbf{m} = m_i dx^i \otimes d^n x$ (regarding $\mathbf{m}$ as a one-form density) and $\mathbf{u} = u^i \partial/\partial x^i$ (regarding $\mathbf{u}$ as a vector field), the EPDiff equations read

$$\frac{\partial}{\partial t} m_i + u^j \frac{\partial m_i}{\partial x^j} + m_j \frac{\partial u^j}{\partial x^i} + m_i \frac{\partial u^j}{\partial x^j} = 0\,. \tag{2.2}$$

The EPDiff equations can also be written very nicely as

$$\frac{\partial \mathbf{m}}{\partial t} + \pounds_{\mathbf{u}} \mathbf{m} = 0, \tag{2.3}$$

where $\pounds_{\mathbf{u}} \mathbf{m}$ denotes the Lie derivative of the momentum one form density $\mathbf{m}$ with respect to the velocity vector field $\mathbf{u}$.

As mentioned earlier, if ℓ is a quadratic function of $\mathbf{u}$, then the EPDiff equation ((2.1) or, equivalently (2.3)), is the Eulerian description of geodesic motion on the diffeomorphism group of the underlying space (in this case $\mathbb{R}^n$). The corresponding metric is the right invariant metric on the group whose value on the Lie algebra (the group's tangent space at the identity—the space of vector fields), is defined by ℓ. Since the Lagrangian ℓ is positive and quadratic in $\mathbf{u}$, the momentum density is linear in $\mathbf{u}$ and so defines a positive symmetric operator $Q_{\rm op}$ by

$$\mathbf{m} = \frac{\delta \ell}{\delta \mathbf{u}} = Q_{\rm op} \mathbf{u}\,.$$

Variational Formulation. Following the variational formulation of EP theory, the particular EP equation (2.1) may be derived from the following constrained variational principle:

$$\delta \int \ell(\mathbf{u})\, dt = 0\,.$$

The variations are constrained to have the form

$$\delta \mathbf{u} = \dot{\mathbf{w}} + \mathbf{w} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{w}.$$

This assertion may of course be verified directly. These constraints are analogous to the so-called "Lin constraints" used for a similar variational principle for fluid mechanics. (See Marsden and Ratiu [1999] for a discussion and references.)

Hamiltonian Formulation. A Legendre transformation yields the Hamiltonian,

$$H(\mathbf{m}) = \langle \mathbf{m}, \mathbf{u} \rangle - \ell(\mathbf{u})\,,$$

where $\langle\, ,\, \rangle$ is the natural pairing between one form densities and vector fields given by integration. This Hamiltonian is the corresponding quadratic form for the momentum,

$$H(\mathbf{m}) = \ell(Q_{\rm op}^{-1}(\mathbf{m}))\,. \tag{2.4}$$

Of course it often happens that $Q_{\rm op}$ is a differential operator and in this case the inverse is usually given in terms of the convolution with the Green's function G, corresponding to the appropriate solution domain and boundary conditions;

$$\mathbf{u} = \frac{\delta H(\mathbf{m})}{\delta \mathbf{m}} = G * \mathbf{m}\,.$$

According to the general theory, the EP equation (2.1) may be expressed in Hamiltonian form by using the Lie-Poisson bracket on $\mathfrak{M}$ as

$$\frac{\partial}{\partial t}\mathbf{m} = \{\mathbf{m}\,,\,H\}_{LP} = -\,\mathrm{ad}^*_{\delta H/\delta \mathbf{m}}\mathbf{m}\,. \tag{2.5}$$

One-dimensional CH Peakon Solutions. We return now to the CH equation (1.2), which, as we have noted, is the same as the EPDiff equation (2.1) for the case of one spatial dimension, when the momentum velocity relationship is defined by the Helmholtz equation, $m = u - \alpha^2 u_{xx}$. In one dimension, the CH equation has solutions whose momentum is supported at points on the real line via the following sum over Dirac delta measures,

$$m(x,t) = \sum_{i=1}^{N} p_i(t)\,\delta\big(x - q_i(t)\big)\,. \tag{2.6}$$

The velocity corresponding to this measure-valued momentum is obtained by convolution with the Green's function,

$$G(|x-y|) = \tfrac{1}{2}e^{-|x-y|/\alpha}\,,$$

for the one-dimensional Helmholtz operator, $Q_{\rm op} = (1 - \alpha^2\partial_x^2)$, appearing in the CH momentum velocity relationship, $m = Q_{\rm op}u$. Consequently, the CH velocity corresponding to this momentum is given by a superposition of peaked traveling wave pulses,

$$u(x,t) = \frac{1}{2}\sum_{i=1}^{N} p_i(t)\,e^{-|x-q_i(t)|/\alpha}\,. \tag{2.7}$$

Thus, the superposition of "peakons" in velocity arises from the delta function solution ansatz (2.6) for the momentum.

Remarkably, the isospectral eigenvalue problem for the CH equation implies that *only* these singular solutions emerge asymptotically in the solution of the initial value problem in one dimension, Camassa and Holm [1993]. Figure 2.1 shows the emergence of peakons from an initially Gaussian velocity distribution and their subsequent elastic collisions in a periodic one-dimensional domain.[1] This figure demonstrates that singular solutions dominate the initial value problem and, thus, that it is imperative to go beyond smooth solutions for the CH equation; the situation is similar for the EPDiff equation.

[1] Figure 2.1 was kindly supplied by Martin Staley

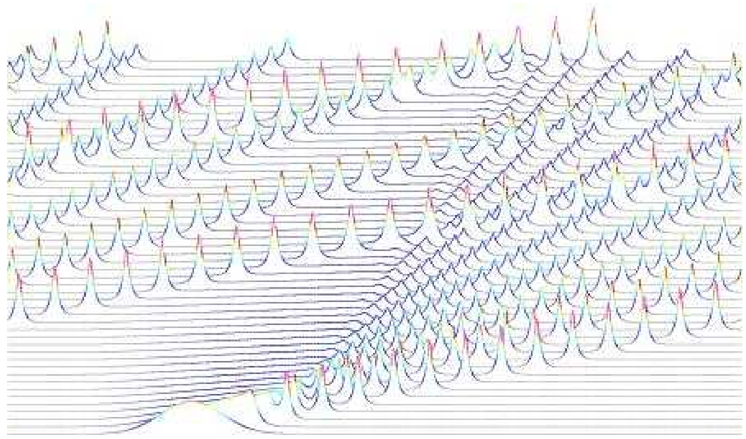

Figure 2.1: This figure shows a smooth localized (Gaussian) initial condition for the CH equation breaking up into an ordered train of peakons as time evolves (the time direction being vertical, which then eventually wrap around the periodic domain and interacting with other slower emergent peakons and causing a phaseshift (c.f. Alber and Marsden [1992]).

Remarkably, the dynamical equations for $p_i(t)$ and $q_i(t)$, $i = 1, \ldots, N$, that arise from solution ansatz (2.6-2.7) comprise an integrable system for any N. This system is studied in (Alber, Camassa, Fedorov, Holm and Marsden [2001]) and references therein. See also Vaninsky [2002, 2003] for discussions of how the integrable dynamical system for N peakons is related to the Toda chain with open ends.

Generalizing the CH peakon solutions to n dimensions Building on the peakon solutions for the CH equation and the pulsons for its generalization to other traveling-wave shapes (see Fringer and Holm [2001]), Holm and Staley [2003] introduced the following measure-valued (that is, density valued) ansatz for the n−dimensional solutions of the EPDiff equation (2.1):

$$\mathbf{m}(\mathbf{x}, t) = \sum_{a=1}^{N} \int \mathbf{P}^a(s, t)\, \delta\big(\, \mathbf{x} - \mathbf{Q}^a(s, t)\,\big) ds. \tag{2.8}$$

These solutions are vector-valued functions supported in $\mathbb{R}^n$ on a set of N surfaces (or curves) of codimension $(n - k)$ for $s \in \mathbb{R}^k$ with $k < n$. They may, for example, be supported on sets of points (vector peakons, $k = 0$), one-dimensional filaments (strings, $k = 1$), or two-dimensional surfaces (sheets, $k = 2$) in three dimensions.

One of the main results of this paper is that the singular solution ansatz (2.8) is a momentum map. This result helps to organize the theory and to suggest new avenues of exploration, as we shall explain.

Substitution of the solution ansatz (2.8) into the EPDiff equations (2.1) implies the following integro-partial-differential equations (IPDEs) for the evolution of such

strings and sheets,

$$\frac{\partial}{\partial t}\mathbf{Q}^a(s,t) = \sum_{b=1}^{N}\int \mathbf{P}^b(s',t)\,G(\mathbf{Q}^a(s,t)-\mathbf{Q}^b(s',t)\big)ds'\,, \tag{2.9}$$
$$\frac{\partial}{\partial t}\mathbf{P}^a(s,t) = -\sum_{b=1}^{N}\int \big(\mathbf{P}^a(s,t)\cdot\mathbf{P}^b(s',t)\big)\,\frac{\partial}{\partial\mathbf{Q}^a(s,t)}G\big(\mathbf{Q}^a(s,t)-\mathbf{Q}^b(s',t)\big)\,ds'\,.$$

Importantly for the interpretation of these solutions, the coordinates $s\in\mathbb{R}^k$ turn out to be Lagrangian coordinates. The velocity field corresponding to the momentum solution ansatz (2.8) is given by

$$\mathbf{u}(\mathbf{x},t) = G*\mathbf{m} = \sum_{b=1}^{N}\int \mathbf{P}^b(s',t)\,G\big(\,\mathbf{x}-\mathbf{Q}^b(s',t)\,\big)ds'\,,\quad \mathbf{u}\in\mathbb{R}^n\,. \tag{2.10}$$

When evaluated along the curve $\mathbf{x}=\mathbf{Q}^a(s,t)$, the velocity satisfies,

$$\mathbf{u}(\mathbf{x},t)\Big|_{\mathbf{x}=\mathbf{Q}^a(s,t)} = \sum_{b=1}^{N}\int \mathbf{P}^b(s',t)\,G\big(\,\mathbf{Q}^a(s,t)-\mathbf{Q}^b(s',t)\,\big)ds' = \frac{\partial\mathbf{Q}^a(s,t)}{\partial t}\,. \tag{2.11}$$

Thus, the lower-dimensional support sets defined on $\mathbf{x}=\mathbf{Q}^a(s,t)$ and parameterized by coordinates $s\in\mathbb{R}^k$ move with the fluid velocity. Moreover, equations (2.9) for the evolution of these support sets are canonical Hamiltonian equations,

$$\frac{\partial}{\partial t}\mathbf{Q}^a(s,t) = \frac{\delta H_N}{\delta\mathbf{P}^a}\,,\qquad \frac{\partial}{\partial t}\mathbf{P}^a(s,t) = -\,\frac{\delta H_N}{\delta\mathbf{Q}^a}\,. \tag{2.12}$$

The Hamiltonian function $H_N : (\mathbb{R}^n\times\mathbb{R}^n)^N\to\mathbb{R}$ is,

$$H_N = \frac{1}{2}\iint\sum_{a\,,\,b=1}^{N}\big(\mathbf{P}^a(s,t)\cdot\mathbf{P}^b(s',t)\big)\,G\big(\mathbf{Q}^a(s,t)-\mathbf{Q}^b(s',t)\big)\,ds\,ds'\,. \tag{2.13}$$

This is the Hamiltonian for geodesic motion on the cotangent bundle of a set of curves $\mathbf{Q}^a(s,t)$ with respect to the metric given by G. This dynamics was investigated numerically in Holm and Staley [2003] to which we refer for more details of the solution properties.

One of our main goals is to show that the solution ansatz (2.8) can be phrased in terms of a momentum map that naturally arises in this problem. This geometric feature underlies the remarkable reduction properties of the EPDiff equation and "explains" why the preceding equations must be Hamiltonian, namely, because momentum maps are Poisson maps.

As explained in general terms in Marsden and Weinstein [1983], the way one implements a coadjoint orbit reduction is through a momentum map, and this holds even for the case of singular orbits (again ignoring functional analytic details). Thus, in summary, *the reduction* (2.12) *is the* EPDiff *analog of the reduction in fluid*

mechanics (that is, the $\mathrm{EPDiff}_{\mathrm{Vol}}$ *equations) to point (or blob) vortex dynamics, vortex filaments, or sheets.*

There are, however, some important differences between vortex dynamics for incompressible flows and the dynamics of the measure valued EPDiff solutions. For example, the Lagrangian representations of the equations of motion show that EPDiff solutions have inertia, while the corresponding solutions for point (or blob) vortices of the $\mathrm{EPDiff}_{\mathrm{Vol}}$ dynamics have no inertia. That is, the equations of motion for measure valued solutions on $\mathrm{EPDiff}_{\mathrm{Vol}}$ are *first order* in time, while the dynamical equations for measure valued solutions on EPDiff are *second order* in time. This difference has profound effects on the properties of the solutions, especially on their stability properties. Numerical investigations of Holm and Staley [2003] show, for example, that the codimension-one solutions of EPDiff are stable, while higher codimension solutions of EPDiff are very unstable to codimension-one perturbations. In contrast, the codimension-two solutions of $\mathrm{EPDiff}_{\mathrm{Vol}}$ are known to be stable.

Comments on the Physical Meaning of the Equations. The EPDiff equations with the Helmholtz relation between velocity and momentum are not quite the CH equations for surface waves in 2D. Those would take precisely the same form, but the shallow water wave relation in the 2D CH approximation would be

$$m = u - \alpha^2 \,\mathrm{Grad}\,\mathrm{Div}\, u; \quad \text{that is,} \quad m_i = u_i - \alpha^2 u_{j,ji}$$

rather than the Helmholtz operator form,

$$m = u - \alpha^2 \,\mathrm{Div}\,\mathrm{Grad}\, u \quad \text{that is,} \quad m_i = u_i - \alpha^2 u_{i,jj}$$

The corresponding Lagrangians are, respectively,

$$l_{\mathrm{CH}}(u) = \frac{1}{2}\int \left(|u|^2 + \alpha^2 (\mathrm{Div}\, u)^2\right) dx\, dy. \tag{2.14}$$

and

$$l_{\mathrm{EPDiff}}(u) = \frac{1}{2}\int \left(|u|^2 + \alpha^2 |\,\mathrm{Grad}\, u|^2\right) dx\, dy. \tag{2.15}$$

This difference was noted in Kruse, Scheurle and Du [2001], which identified (2.14) as the generalization of (1.4) for water waves in two dimensions. One may also verify this by considering the limit of the Green-Nagdhi equations for small potential energy. (The CH equation arises in this limit. The Lake and Great Lake equations of Camassa, Holm and Levermore [1996, 1997] also arise in a variant of this limit.)

Remarkably, the numerics in Holm and Staley [2003] show that the solutions for a variety of initial conditions are indistinguishable in these two cases. The initial conditions in Holm and Staley [2003] were all spatially confined velocity distributions.

Notice that this difference affects the choice of Hamiltonian, but the equations are still Euler-Poincaré equations for the diffeomorphism group and the description

of the anzatz (2.8) as a momentum map is independent of this difference in the equations.

Figure 2.2 shows the striking reconnection phenomenon seen in the nonlinear interaction between wave-trains, as simulated by numerical solutions of the EPDiff equation and observed for internal waves in the Ocean. Fig 2.2(a) shows a frame taken from simulations of the initial value problem for the EPDiff equation in two dimensions, excerpted from Holm and Staley [2003]. (See also Holm, Putkaradze and Stechmann [2003].)

Fig 2.2(b) shows the interaction of two internal wave trains propagating at the interface of different density levels in the South China Sea, as observed from the Space Shuttle using synthetic aperture radar, courtesy of A. Liu (2002). Importantly, both Fig 2.2(a) and Fig 2.2(b) show nonlinear reconnection occurring in the wave train interaction as their characteristic feature.

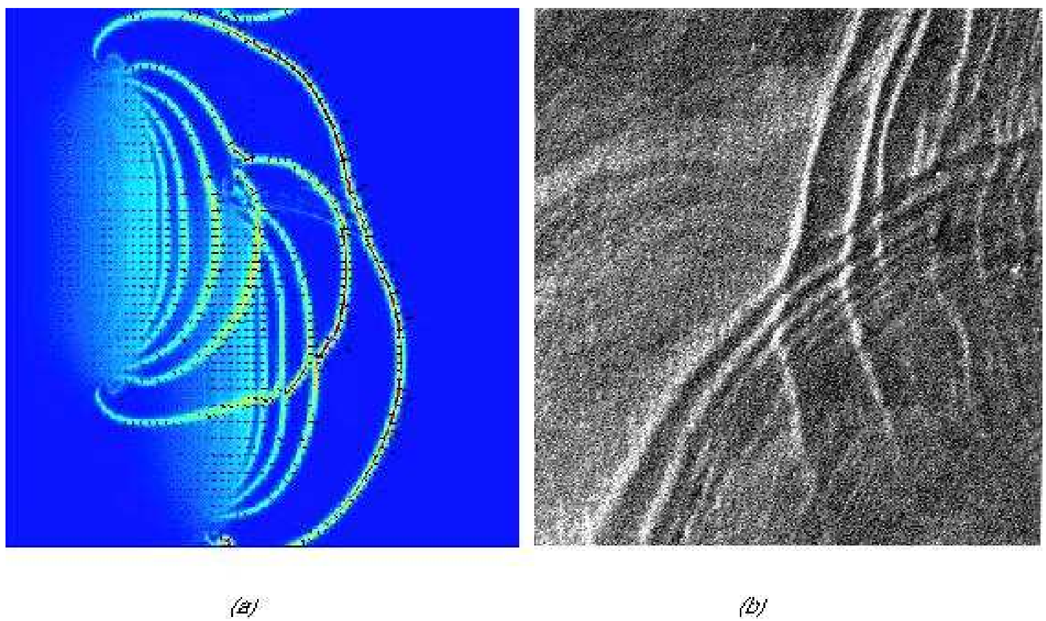

Figure 2.2: Comparison of evolutionary EPDiff solutions in two dimensions (a) and Synthetic Aperture Radar observations by the Space Shuttle of internal waves in the South China Sea (b). Both Figures show nonlinear reconnection occurring in the wave train interaction as their characteristic feature.

Internal waves and these sorts of interactions are generally thought to be described by the KP equation, and so any relations among the KP equation, the EPDiff equation and the 2D CH equation would be of great interest to explore; cf. Liu et al. [1998]. The derivations of the KP equation and the CH equation differ in the way the transverse motions are treated in the asymptotics; so some difference in their solution behavior is to be expected.

3 Smoothness of the Lagrangian Equations

The One-Dimensional Case. Based on a formal argument given in Holm, Marsden and Ratiu [1998a], it was shown in Shkoller [1998] that when the CH equation (1.2) is transformed to Lagrangian variables, it defines a smooth vector field (that is, one obtains an evolution equation with no derivative loss). This means that one can show using ODE methods that the initial value problem is well-posed and one may also establish other important properties of the equations when the data is sufficiently smooth.

As above, we write the relation between m and u as $m = Q_{\text{op}}u$, so that in the one dimensional case, Q_{op} is the operator $Q_{\text{op}} = \text{Id} - \alpha^2\partial_{xx}$. We first recall how the equations are transformed into Lagrangian variables. Introduce the one parameter curve of diffeomorphisms $\eta(x_0, t)$ defined implicitly by

$$\frac{\partial}{\partial t}\eta(x_0, t) = u(\eta(x_0, t), t), \tag{3.1}$$

so that η is a geodesic in the group of diffeomorphisms of $\mathbb{R}$ (or, with periodic boundary conditions, of the circle S^1) equipped with the right invariant metric equaling the H^1 metric at the identity.

We compute the second time derivative of η in a straightforward way by differentiating (3.1) using the chain rule:

$$\frac{\partial^2\eta}{\partial t^2} = uu_x + \frac{\partial u}{\partial t}\,.$$

Acting on this equation with Q_{op} and using the definition $m = Q_{\text{op}}u$ yields

$$\begin{aligned} Q_{\text{op}}\frac{\partial^2\eta}{\partial t^2} &= Q_{\text{op}}(uu_x) - u\partial_x(Q_{\text{op}}u) + um_x + \frac{\partial m}{\partial t} \\ &= [Q_{\text{op}}\,,\, u\partial_x]u - 2mu_x \\ &= -3\alpha^2 u_x u_{xx} - 2mu_x \\ &= -\alpha^2 u_x u_{xx} - 2uu_x\,, \end{aligned}$$

where the third step uses the commutator relation calculated from the product rule,

$$[Q_{\text{op}}\,,\, u\partial_x]u = -3\alpha^2 u_x u_{xx}\,.$$

Hence, the preceding equation becomes

$$\frac{\partial^2\eta}{\partial t^2} = -\,\frac{1}{2}\,Q_{\text{op}}^{-1}\,\partial_x\left(\alpha^2 u_x^2 + 2u^2\right). \tag{3.2}$$

The important point about this equation is that the right hand side has no derivative loss. That is, if u is in the Sobolev space H^s for $s > 5/2$, then the right hand is also in the same space. Regarding the right hand side as a function of η and $\partial\eta/\partial t$, we see that it is plausible that the second order evolution equation (3.2) for η defines a smooth ODE on the group of H^s diffeomorphisms. (This argument requires the use of, for example, weighted Sobolev spaces in the case $x \in \mathbb{R}$).

The above is the essence of the argument given in Shkoller [1998], Remark 3.5, which in turn makes use of the type of arguments found in Ebin and Marsden [1970] for the incompressible case and which shows, by a more careful argument, that the spray is smooth if $s > 3/2$. However, one should note that the complete argument is not quite so simple (just as in the case of incompressible fluids). A subtilty arises because smoothness means as a function of $\eta\,,\dot\eta$. Hence, one must express u in terms of η, namely through the relation $u_t = \dot\eta_t \circ \eta_t^{-1}$, where the subscript t here denotes that this argument is held fixed, and is not a partial derivative. Doing this, one sees that, while there is clearly no derivative loss, the right hand side of (3.2) does involve η_t^{-1} and the map $\eta_t \mapsto \eta_t^{-1}$ is known to *not* be smooth (just as in Ebin and Marsden [1970]). Nevertheless, the *combination* that appears in (3.2) is, quite remarkably, a smooth function of $\eta, \dot\eta$.

The n-Dimensional Case. The above argument readily generalizes to the n-dimensions, which we shall present in the case of $\mathbb{R}^n$ or the flat n-torus $\mathbb{T}^n$ for simplicity. Namely, we still have the relation

$$\frac{\partial}{\partial t}\eta(\mathbf{x}_0,t) = \mathbf{u}(\eta(\mathbf{x}_0,t),t),$$

between η and $\mathbf{u}$. Consequently, we may compute the second partial time derivative of η in the usual fashion using the chain rule:

$$\frac{\partial^2\eta}{\partial t^2} = \mathbf{u}\cdot\nabla\mathbf{u} + \frac{\partial\mathbf{u}}{\partial t}\,.$$

Therefore, as in the one dimensional case, we get

$$Q_{\rm op}\frac{\partial^2\eta}{\partial t^2} = \big[Q_{\rm op}\,,\,(\mathbf{u}\cdot\nabla)\big]\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{m} + \frac{\partial\mathbf{m}}{\partial t}\,.$$

Calculating the commutator relation in n-dimensions gives

$$\big[Q_{\rm op}\,,\,(\mathbf{u}\cdot\nabla)\big]\mathbf{u} = -\,\alpha^2{\rm div}\Big(\nabla\mathbf{u}\cdot\nabla\mathbf{u} + \nabla\mathbf{u}\cdot\nabla\mathbf{u}^T\Big) + \alpha^2(\nabla\mathbf{u})\cdot\nabla{\rm div}\,\mathbf{u}$$

or, in components,

$$\Big(\big[Q_{\rm op}\,,\,(\mathbf{u}\cdot\nabla)\big]\mathbf{u}\Big)_i = -\,\alpha^2\partial_k\Big(u_{i,j}\,u_{j,k} + u_{i,j}\,u_{k,j}\Big) + \alpha^2(u_{i,j})\,\partial_j{\rm div}\,\mathbf{u}\,, \tag{3.3}$$

with a sum on repeated indices.

Upon substituting the preceding commutator relation, the EPDiff equation (2.1) and the vector calculus identity

$$-\nabla\mathbf{u}^T\cdot\mathbf{m} = \alpha^2\,{\rm div}\,\Big(\nabla\mathbf{u}^T\cdot\nabla\mathbf{u}\Big) - \nabla\Big(\frac{1}{2}|\mathbf{u}|^2 + \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2\Big)\,, \tag{3.4}$$

then imply the $n-$dimensional result

$$\begin{aligned} Q_{\rm op}\frac{\partial^2\eta}{\partial t^2} &= -\,\alpha^2{\rm div}\Big(\nabla\mathbf{u}\cdot\nabla\mathbf{u} + \nabla\mathbf{u}\cdot\nabla\mathbf{u}^T - \nabla\mathbf{u}^T\cdot\nabla\mathbf{u} - \nabla\mathbf{u}^T({\rm div}\,\mathbf{u}) + \frac{1}{2}\,{\rm Id}|\nabla\mathbf{u}|^2\Big) \\ &\quad -\mathbf{u}({\rm div}\,\mathbf{u}) - \frac{1}{2}\,\nabla|\mathbf{u}|^2\,. \end{aligned}$$

This form of the EPDiff equation is useful for interpreting some of its solution behavior. As in the one dimensional case, the crucial point is that the right hand side involves at most second derivatives of $\mathbf{u}$; so there is no derivative loss in the overall expression. The precise formulation of the smoothness result thus should hold, as in the one-dimensional case, although, naturally the details are a bit more complicated. We shall leave the technical details for another publication.

A consequence is that the equations are well posed *for short time* if the initial data is smooth enough and, hence, e.g., two nearby smooth solutions can be joined by a unique geodesic. Moreover, the solutions of the EPDiff equation are automatically C^∞ in time. In the sections that follow, we will be interested in nonsmooth data, which is in stark contrast to the preceding discussion, which requires initial data that is at least C^1.

Remarkably, the same smoothness results hold for the case of the LAE-α (Lagrangian averaged Euler) equations, a set of incompressible equations in which small scale fluctuations are averaged. One can view the LAE-α equations as the incompressible version of the EPDiff equations. This smoothness property for the LAE-α equations was shown by Shkoller [1998] for regions with no boundary and for regions with boundary (for various boundary conditions), it was shown in Marsden, Ratiu and Shkoller [2000]. However, unlike the incompressible case, the results apparently do not hold if α is zero (as also noted in Shkoller [1998]). This sort of smoothness result also appears not to hold for many other equations, such as the KdV equation, even though it too can be realized as Euler-Poincaré equations on a Lie algebra, or as geodesics on a group, in this case the Bott-Virasoro group, as explained in Marsden and Ratiu [1999] and references therein.

The Development of Singularities. The smoothness property just discussed does not preclude the development in finite time of singular solutions from smooth localized initial data as was indicated in Figure 2.1. To capture the local singularities in the EPDiff solution (either verticality in slope, or discontinuities in its spatial derivative) that develop in finite time from arbitrarily smooth initial conditions, one must enlarge the solution class of interest, by considering weak solutions.

There are a number of papers on weak solutions of the CH equation such as Xin and Zhang [2000] that we will not survey here. We just mention that the theory is not yet complete, as it is still unknown in what sense one may define *global unique* weak solutions to the CH equations in H^1. As discussed in Alber, Camassa, Fedorov, Holm and Marsden [2001] for the CH equation, one most likely must consider weak solutions in the *spacetime sense*.

The steepening lemma of Camassa and Holm [1993] proves that in one dimension any initial velocity distribution whose spatial profile has an inflection point with negative slope (for example, any antisymmetric smooth initial distribution of velocity on the real line) will develop a vertical slope in finite time. Note that the peakon solution (2.7) has no inflection points, so it is not subject to the steepening lemma. However, the steepening lemma underlies the mechanism for forming these singular solutions, which are continuous but have discontinuous spatial derivatives;

they also lie in H^1 and have finite energy. We conclude that solutions with initial conditions in H^s with $s > (n/2) + 1$ go to infinity in the H^s norm in finite time, but remain in H^1 and presumably continue to exist in a weak spacetime sense for all time in H^1.

Numerical evidence in higher dimensions and the inverse scattering solution for the CH equation in one dimension (the latter has *only* discrete eigenvalues, corresponding to peakons) both show that the singular solutions completely dominate the time-asymptotic dynamics of the initial value problem (IVP). This singular IVP behavior is one of the main discoveries of Camassa and Holm [1993]. This singular behavior has drawn a great deal of mathematical interest to the CH equation and its relatives, such as EPDiff. The other properties of CH — its complete integrability, inverse scattering transform, connections to algebraic geometry and elliptical billiards, bi-Hamiltonian structure, etc. — are all interesting, too. However, the requirement of dealing with singularity as its main solution phenomenon is the primary aspect of CH (and EPDiff). We aim to show that many of the properties of these singular solutions of CH and EPDiff are captured by recognizing that the singular solution ansatz itself is a momentum map. This momentum map property explains, for example, why the singular solutions (2.8) form an invariant manifold for any value of N.

In one dimension, the complete integrability of the CH equation as a Hamiltonian system and its soliton paradigm completely explain the emergence of peakons in the CH dynamics. Namely, their emergence reveals the initial condition's soliton (peakon) content. However, beyond one dimension, we do not have an explicit mechanism for explaining why *only* singular solution behavior emerges in numerical simulations. One hopes that eventually a theory will be developed for explaining this singular solution phenomenon in higher dimensions. Such a theory might, for example, parallel the well-known explanation of the formation of shocks for hyperbolic partial differential equations. (Note, however, that EPDiff is not hyperbolic, because the relation $\mathbf{u} = G * \mathbf{m}$ between its velocity and momentum is nonlocal.)

In the remainder of this work, we shall focus our attention on the momentum map properties of the invariant manifold of singular solutions (2.8) of the EPDiff equation.

4 The Singular Solution Momentum Map

The Momentum Ansatz (2.8) is a Momentum Map. The purpose of this section is to show that the solution ansatz (2.8) for the momentum vector in the EPDiff equation (2.1) defines a momentum map for the action of the group of diffeomorphisms on the support sets of the Dirac delta functions. These support sets are the analogs of points on the real line for the CH equation in one dimension. They are points, curves, or surfaces in $\mathbb{R}^n$ for the EPDiff equation in $n-$dimensions.

This result, as we shall discuss in greater detail later, shows that the solution ansatz (2.8) fits naturally into the scheme of Clebsch, or canonical variables in the

sense advocated by Marsden and Weinstein [1983] as well as showing that these singular solutions evolve on special coadjoint orbits for the diffeomorphism group.

One can summarize by saying that the map that implements the canonical $(\mathbf{Q}, \mathbf{P})$ variables in terms of singular solutions is a (cotangent bundle) momentum map. Such momentum maps are Poisson maps; so the canonical Hamiltonian nature of the dynamical equations for $(\mathbf{Q}, \mathbf{P})$ fits into a general theory which also provides a framework for suggesting other avenues of investigation.

Theorem 4.1. *The momentum ansatz* (2.8) *for measure-valued solutions of the* EPDiff *equation* (2.1)*, defines an equivariant momentum map*

$$\mathbf{J}_{\text{Sing}} : T^* \operatorname{Emb}(S, \mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n)^*$$

that we will call the ***singular solution momentum map****.*

We shall explain the notation used in this statement in the course of the proof. Right away, however, we note that the sense of "defines" is that expressing $\mathbf{m}$ in terms of $\mathbf{Q}, \mathbf{P}$ (which are, in turn, functions of s) can be regarded as a map from the space of $(\mathbf{Q}(s), \mathbf{P}(s))$ to the space of $\mathbf{m}$'s. This will turn out to be the Lagrange-to-Euler map for the fluid description of the singular solutions.

We shall give two proofs of this result from two rather different points of view. The first proof below uses the formula for a momentum map for a cotangent lifted action, while the second proof focuses on a Poisson bracket computation. Each proof also explains the context in which one has a momentum map. (See Marsden and Ratiu [1999] for general background on momentum maps.)

First Proof. For simplicity and without loss of generality, let us take $N = 1$ and so suppress the index a. That is, we shall take the case of an isolated singular solution. As the proof will show, this is not a real restriction.

To set the notation, fix a k-dimensional manifold S with a given volume element and whose points are denoted $s \in S$. Let $\operatorname{Emb}(S, \mathbb{R}^n)$ denote the set of smooth embeddings $\mathbf{Q} : S \to \mathbb{R}^n$. (If the EPDiff equations are taken on a manifold M, replace $\mathbb{R}^n$ with M.) Under appropriate technical conditions, which we shall just treat formally here, $\operatorname{Emb}(S, \mathbb{R}^n)$ is a smooth manifold. (See, for example, Ebin and Marsden [1970] and Marsden and Hughes [1983] for a discussion and references.)

The tangent space $T_{\mathbf{Q}} \operatorname{Emb}(S, \mathbb{R}^n)$ to $\operatorname{Emb}(S, \mathbb{R}^n)$ at the point $\mathbf{Q} \in \operatorname{Emb}(S, \mathbb{R}^n)$ is given by the space of ***material velocity fields***, namely the linear space of maps $\mathbf{V} : S \to \mathbb{R}^n$ that are vector fields over the map $\mathbf{Q}$. The dual space to this space will be identified with the space of one-form densities over $\mathbf{Q}$, which we shall regard as maps $\mathbf{P} : S \to (\mathbb{R}^n)^*$. In summary, the cotangent bundle $T^* \operatorname{Emb}(S, \mathbb{R}^n)$ is identified with the space of pairs of maps $(\mathbf{Q}, \mathbf{P})$.

These give us the domain space for the singular solution momentum map. Now we consider the action of the symmetry group. Consider the group $\mathfrak{G} = \text{Diff}$ of diffeomorphisms of the space $\mathfrak{S}$ in which the EPDiff equations are operating, concretely in our case $\mathbb{R}^n$. Let it act on $\mathfrak{S}$ by composition on the *left*. Namely for $\eta \in \operatorname{Diff}(\mathbb{R}^n)$, we let

$$\eta \cdot \mathbf{Q} = \eta \circ \mathbf{Q}. \tag{4.1}$$

Now lift this action to the cotangent bundle $T^* \operatorname{Emb}(S, \mathbb{R}^n)$ in the standard way (see, for instance, Marsden and Ratiu [1999] for this construction). This lifted action is a symplectic (and hence Poisson) action and has an equivariant momentum map. *We claim that this momentum map is precisely given by the ansatz* (2.8).

To see this, we just need to recall and then apply the general formula for the momentum map associated with an action of a general Lie group $\mathfrak{G}$ on a configuration manifold Q and cotangent lifted to T^*Q.

First let us recall the general formula. Namely, the momentum map is the map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ ($\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$) defined by

$$\mathbf{J}(\alpha_q) \cdot \xi = \langle \alpha_q, \xi_Q(q) \rangle \,, \tag{4.2}$$

where $\alpha_q \in T_q^* Q$ and $\xi \in \mathfrak{g}$, where ξ_Q is the infinitesimal generator of the action of $\mathfrak{G}$ on Q associated to the Lie algebra element ξ, and where $\langle \alpha_q, \xi_Q(q) \rangle$ is the natural pairing of an element of $T_q^* Q$ with an element of $T_q Q$.

Now we apply this formula to the special case in which the group $\mathfrak{G}$ is the diffeomorphism group $\operatorname{Diff}(\mathbb{R}^n)$, the manifold Q is $\operatorname{Emb}(S, \mathbb{R}^n)$ and where the action of the group on $\operatorname{Emb}(S, \mathbb{R}^n)$ is given by (4.1). The sense in which the Lie algebra of $\mathfrak{G} = \operatorname{Diff}$ is the space $\mathfrak{g} = \mathfrak{X}$ of vector fields is well-understood. Hence, its dual is naturally regarded as the space of one-form densities. The momentum map is thus a map $\mathbf{J} : T^* \operatorname{Emb}(S, \mathbb{R}^n) \to \mathfrak{X}^*$.

With $\mathbf{J}$ given by (4.2), we just need to work out this formula. First, we shall work out the infinitesimal generators. Let $X \in \mathfrak{X}$ be a Lie algebra element. By differentiating the action (4.1) with respect to η in the direction of X at the identity element we find that the infinitesimal generator is given by

$$X_{\operatorname{Emb}(S, \mathbb{R}^n)}(\mathbf{Q}) = X \circ \mathbf{Q}.$$

Thus, taking α_q to be the cotangent vector $(\mathbf{Q}, \mathbf{P})$, equation (4.2) gives

$$\begin{aligned} \langle \mathbf{J}(\mathbf{Q}, \mathbf{P}), X \rangle &= \langle (\mathbf{Q}, \mathbf{P}), X \circ \mathbf{Q} \rangle \\ &= \int_S P_i(s) X^i(\mathbf{Q}(s)) d^k s. \end{aligned}$$

On the other hand, note that the right hand side of (2.8) (again with the index a suppressed, and with t suppressed as well), when paired with the Lie algebra element X is

$$\begin{aligned} \left\langle \int_S \mathbf{P}(s)\, \delta\left(\mathbf{x} - \mathbf{Q}(s)\right) d^k s, X \right\rangle &= \int_{\mathbb{R}^n} \int_S \Big(P_i(s)\, \delta\left(\mathbf{x} - \mathbf{Q}(s)\right) d^k s \Big) X^i(\mathbf{x}) d^n x \\ &= \int_S P_i(s) X^i(\mathbf{Q}(s) d^k s. \end{aligned}$$

This shows that the expression given by (2.8) is equal to $\mathbf{J}$ and so the result is proved. ■

Second Proof. As is standard (see, for example, Marsden and Ratiu [1999]), one can characterize momentum maps by means of the following relation, required to hold for all functions F on $T^*\operatorname{Emb}(S,\mathbb{R}^n)$; that is, functions of $\mathbf{Q}$ and $\mathbf{P}$:

$$\{F, \langle \mathbf{J}, \xi \rangle\} = \xi_P[F]\,. \tag{4.3}$$

In our case, we shall take $\mathbf{J}$ to be given by the solution ansatz and verify that it satisfies this relation. To do so, let $\xi \in \mathfrak{X}$ so that the left side of (4.3) becomes

$$\left\{F, \int_S P_i(s)\xi^{\,i}(\mathbf{Q}(s))d^{\,k}s\right\} = \int_S \left[\frac{\delta F}{\delta Q^i}\xi^{\,i}(\mathbf{Q}(s)) - P_i(s)\frac{\delta F}{\delta P_j}\frac{\delta}{\delta Q^j}\xi^{\,i}(\mathbf{Q}(s))\right] d^{\,k}s\,.$$

On the other hand, one can directly compute from the definitions that the infinitesimal generator of the action on the space $T^*\operatorname{Emb}(S,\mathbb{R}^n)$ corresponding to the vector field $\xi^i(\mathbf{x})\frac{\partial}{\partial Q^i}$ (a Lie algebra element), is given by (see Marsden and Ratiu [1999], formula (12.1.14)):

$$\delta\mathbf{Q} = \xi \circ \mathbf{Q}\,, \quad \delta\mathbf{P} = -\,P_i(s)\frac{\partial}{\partial \mathbf{Q}}\xi^{\,i}(\mathbf{Q}(s)),$$

which verifies that (4.3) holds.

An important element left out in this proof so far is that it does not make clear that the momentum map is *equivariant*, a condition needed for the momentum map to be Poisson. The first proof took care of this automatically since momentum maps for cotangent lifted actions are always equivariant and hence Poisson.

Thus, to complete the second proof, we need to check directly that the momentum map is equivariant. Actually, we shall only check that it is infinitesimally invariant by showing that it is a Poisson map from $T^*\operatorname{Emb}(S,\mathbb{R}^n)$ to the space of $\mathbf{m}$'s (the dual of the Lie algebra of $\mathfrak{X}$) with its Lie-Poisson bracket. This sort of approach to characterize equivariant momentum maps is discussed in an interesting way in Weinstein [2002].

The following computation accomplishes this methodology by showing that the singular solution momentum map is Poisson.

Indeed, we use the canonical Poisson brackets for $\{\mathbf{P}\}$, $\{\mathbf{Q}\}$ and apply the chain rule to compute $\big\{m_i(\mathbf{x}), m_j(\mathbf{y})\big\}$, with notation $\delta\,'_k(\mathbf{y}) \equiv \partial\delta(\mathbf{y})/\partial y^k$. We get

$$\begin{aligned}
&\big\{m_i(\mathbf{x}), m_j(\mathbf{y})\big\} \\
&\quad= \left\{ \sum_{a=1}^N \int ds\, P_i^a(s,t)\,\delta(\mathbf{x}-\mathbf{Q}^a(s,t))\,,\, \sum_{b=1}^N \int ds'\, P_j^b(s',t)\,\delta(\mathbf{y}-\mathbf{Q}^b(s',t)) \right\} \\
&\quad= \sum_{a,b=1}^N \iint ds ds' \Big[\{P_i^a(s), P_j^b(s')\}\,\delta(\mathbf{x}-\mathbf{Q}^a(s))\,\delta(\mathbf{y}-\mathbf{Q}^b(s')) \\
&\qquad - \{P_i^a(s), Q_k^b(s')\}P_j^b(s')\,\delta(\mathbf{x}-\mathbf{Q}^a(s))\,\delta\,'_k(\mathbf{y}-\mathbf{Q}^b(s')) \\
&\qquad - \{Q_k^a(s), P_j^b(s')\}P_i^a(s)\,\delta\,'_k(\mathbf{x}-\mathbf{Q}^a(s))\,\delta(\mathbf{y}-\mathbf{Q}^b(s')) \\
&\qquad + \{Q_k^a(s), Q_\ell^b(s')\}P_i^a(s)P_j^b(s')\,\delta\,'_k(\mathbf{x}-\mathbf{Q}^a(s))\,\delta\,'_\ell(\mathbf{y}-\mathbf{Q}^b(s')) \Big]\,.
\end{aligned}$$

Substituting the canonical Poisson bracket relations

$$
\begin{aligned}
\{P_i^a(s), P_j^b(s')\} &= 0 \\
\{Q_k^a(s), Q_\ell^b(s')\} &= 0, \quad \text{and} \\
\{Q_k^a(s), P_j^b(s')\} &= \delta^{ab}\delta_{kj}\delta(s-s')
\end{aligned}
$$

into the preceding computation yields,

$$
\begin{aligned}
&\big\{m_i(\mathbf{x}), m_j(\mathbf{y})\big\} \\
&= \quad \Big\{ \sum_{a=1}^{N} \int ds P_i^a(s,t)\,\delta(\mathbf{x}-\mathbf{Q}^a(s,t))\,,\, \sum_{b=1}^{N} \int ds' P_j^b(s',t)\,\delta(\mathbf{y}-\mathbf{Q}^b(s',t)) \Big\} \\
&= \quad \sum_{a=1}^{N} \int ds P_j^a(s)\,\delta(\mathbf{x}-\mathbf{Q}^a(s))\,\delta\,'_i(\mathbf{y}-\mathbf{Q}^a(s)) \\
&\qquad - \sum_{a=1}^{N} \int ds P_i^a(s)\,\delta\,'_j(\mathbf{x}-\mathbf{Q}^a(s))\,\delta(\mathbf{y}-\mathbf{Q}^a(s)) \\
&= \quad -\Big(m_j(\mathbf{x}) \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^j}\, m_i(\mathbf{x}) \Big) \delta(\mathbf{x}-\mathbf{y})\,.
\end{aligned}
$$

Thus,

$$
\big\{m_i(\mathbf{x})\,,\, m_j(\mathbf{y})\big\} = -\Big(m_j(\mathbf{x}) \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^j}\, m_i(\mathbf{x}) \Big) \delta(\mathbf{x}-\mathbf{y})\,, \tag{4.4}
$$

which is readily checked to be the Lie-Poisson bracket on the space of m's. This completes the second proof of theorem. ■

Each of these proofs has shown the following basic fact.

Corollary 4.2. *The singular solution momentum map defined by the singular solution ansatz, namely,*

$$
\mathbf{J}_{\text{Sing}} : T^* \operatorname{Emb}(S, \mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n)^*
$$

is a Poisson map from the canonical Poisson structure on $T^* \operatorname{Emb}(S, \mathbb{R}^n)$ to the Lie-Poisson structure on $\mathfrak{X}(\mathbb{R}^n)^*$.

This is perhaps the most basic property of the singular solution momentum map. Some of its more sophisticated properties are outlined in the following section.

Pulling Back the Equations. Since the solution ansatz (2.8) has been shown in the preceding Corollary to be a Poisson map, the pull back of the Hamiltonian from $\mathfrak{X}^*$ to $T^* \operatorname{Emb}(S, \mathbb{R}^n)$ gives equations of motion on the latter space that project to the equations on $\mathfrak{X}^*$. This is why the functions $\mathbf{Q}^a(s,t)$ and $\mathbf{P}^a(s,t)$ satisfy canonical Hamiltonian equations. Note that the coordinate $s \in \mathbb{R}^k$ that labels these functions is a Lagrangian coordinate.

In terms of the pairing

$$\langle\cdot\,,\,\cdot\rangle\,:\,\mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}\,, \tag{4.5}$$

between the Lie algebra $\mathfrak{g}$ (vector fields in $\mathbb{R}^n$) and its dual $\mathfrak{g}^*$ (one-form densities in $\mathbb{R}^n$), the following relation holds for measure-valued solutions under the momentum map (2.8),

$$\begin{aligned}
\langle\mathbf{m}\,,\,\mathbf{u}\rangle &= \int \mathbf{m}\,\cdot\,\mathbf{u}\,d^n\mathbf{x}\,, \quad L^2 \text{ pairing for } \mathbf{m}\,\&\,\mathbf{u}\in\mathbb{R}^n,\\
&= \iint\sum_{a\,,\,b=1}^{N}\big(\mathbf{P}^a(s,t)\cdot\mathbf{P}^b(s',t)\big)\,G\big(\mathbf{Q}^a(s,t)-\mathbf{Q}^b(s',t)\big)\,ds\,ds'\\
&= \int\sum_{a=1}^{N}\mathbf{P}^a(s,t)\cdot\frac{\partial\mathbf{Q}^a(s,t)}{\partial t}\,ds\\
&\equiv \langle\!\langle\mathbf{P}\,,\,\dot{\mathbf{Q}}\rangle\!\rangle,
\end{aligned} \tag{4.6}$$

which is the natural pairing between the points $(\mathbf{Q},\mathbf{P})\in T^*\,\mathrm{Emb}(S,\mathbb{R}^n)$ and $(\mathbf{Q},\dot{\mathbf{Q}})\in T\,\mathrm{Emb}(S,\mathbb{R}^n)$.

The pull-back of the Hamiltonian $H[\mathbf{m}]$ defined on the dual of the Lie algebra $\mathfrak{g}^*$, to $T^*\,\mathrm{Emb}(S,\mathbb{R}^n)$ is easily seen to be consistent with what we had before:

$$H[\mathbf{m}]\equiv\frac{1}{2}\langle\mathbf{m}\,,\,G*\mathbf{m}\rangle=\frac{1}{2}\langle\!\langle\mathbf{P}\,,\,G*\mathbf{P}\rangle\!\rangle\equiv H_N[\mathbf{P},\mathbf{Q}]\,. \tag{4.7}$$

In summary, in concert with the Poisson nature of the singular solution momentum map, we see that the singular solutions in terms of $\mathbf{Q}$ and $\mathbf{P}$ satisfy Hamiltonian equations and also define an invariant solution set for the EPDiff equations. In fact, this invariant solution set is a special coadjoint orbit for the diffeomorphism group, as we shall discuss in the next section.

Smoothness. It would be extremely interesting if the smoothness properties explored in §3 were also valid on the space $T^*\,\mathrm{Emb}(S,\mathbb{R}^n)$. We hope to explore this point in future publications.

5 The Geometry of the Momentum Map

In this section we explore the geometry of the singular solution momentum map discussed in §4 in a little more detail. The main idea may be stated as follows: *simply apply all of the ideas given in Marsden and Weinstein [1983] in a systematic way to the current setting.*

Coadjoint Orbits. The first claim is that *the image of the singular solution momentum map is a coadjoint orbit in $\mathfrak{X}^*$.* This means that (perhaps modulo some issues of connectedness and smoothness, which we do not consider here) the solution anzatz given by (2.8) defines a coadjoint orbit in the space of all one-form densities, regarded as the dual of the Lie algebra of the diffeomorphism group.

These coadjoint orbits should be thought of as singular orbits—that is, due to their special nature, they are not generic. Also, this explains why the singular solutions form dynamically invariant sets—it is because they are coadjoint orbits, which are *symplectic submanifolds* of the Lie-Poisson manifold $\mathfrak{X}(\mathbb{R}^n)^*$.

The idea of the proof of this is simply this: whenever one has an equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$ for the action of a group G on a symplectic or Poisson manifold P, and that action is transitive, then the image of $\mathbf{J}$ is an orbit (or at least a piece of an orbit). This general result, due to Kostant, is stated more precisely in Marsden and Ratiu [1999], Theorem 14.4.5. Roughly speaking, the reason that transitivity holds in our case is because one can "move the manifolds S around at will" using diffeomorphisms.

Symplectic Structure on Orbits. Recall (from, for example, Marsden and Ratiu [1999]), the general formula for the symplectic structure on coadjoint orbits:

$$\Omega_\mu \left(\xi_{\mathfrak{g}^*}(\mu), \eta_{\mathfrak{g}^*}(\mu)\right) = \langle \mu, [\xi, \eta] \rangle \, , \tag{5.1}$$

where $\mu \in \mathfrak{g}^*$ is a chosen point on an orbit and where ξ, η are elements of $\mathfrak{g}$. We use a plus sign in this formula since we are dealing with orbits for the *right action*.

Just as in Marsden and Weinstein [1983], this leads to an explicit formula for the coadjoint orbit symplectic structure in the case of Diff. In the present case, it is a particularly simple and transparent formula.

Recall that in the case of incompressible fluid mechanics, this procedure leads naturally to the symplectic (and Poisson) structure for many interesting singular coadjoint orbits, such as point vortices in the plane, vortex patches, vortex blobs (closely related to the planar LAE-α equations) and vortex filaments.

For the case of the diffeomorphism group, let $\mathcal{O}_{\mathbf{m}}$ denote the coadjoint orbit through the point $\mathbf{m} \in \mathfrak{X}^*(\mathbb{R}^n)$.

Theorem 5.1. *The symplectic structure $\Omega_{\mathbf{m}}$ on $T_{\mathbf{m}}\mathcal{O}_{\mathbf{m}}$ is given by*

$$\Omega_{\mathbf{m}}(\pounds_{u_1}\mathbf{m}, \pounds_{u_2}\mathbf{m}) = -\int \langle \mathbf{m}, [u_1, u_2] \rangle \; d^n x.$$

Proof. We simply plug into the general Kirillov-Kostant-Souriau formula (5.1) for the symplectic structure on coadjoint orbits. (As noted above, there is a $+$ sign, since we are dealing with a *right* invariant system). The only thing needing explanation is that our Lie algebra convention always uses the left Lie bracket. For Diff, this is the *negative* of the usual Lie bracket as is explained in Marsden and Ratiu [1999]. ■

Dual Pairs. The singular solution momentum map $\mathbf{J}_{\text{Sing}} : T^*\operatorname{Emb}(S, \mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n)^*$ forms one leg of a dual pair. The point is that there is another group that acts on $\operatorname{Emb}(S, \mathbb{R}^n)$, namely the group $\operatorname{Diff}(S)$ of diffeomorphisms of S, which acts on the *right*, while $\operatorname{Diff}(\mathbb{R}^n)$ acted on it by composition on the *left* (and which gave rise to our singular solution momentum map). This momentum map will be

denoted $\mathbf{J}_S : T^* \operatorname{Emb}(S, \mathbb{R}^n) \to \mathfrak{X}(S)^*$. The dual pair arises for reasons that are very similar to those in Marsden and Weinstein [1983]. See the following diagram.

$$\begin{tikzcd}
 & T^* \operatorname{Emb}(S, \mathbb{R}^n) \arrow[dl, "\mathbf{J}_{\mathrm{Sing}}"'] \arrow[dr, "\mathbf{J}_S"] & \\
\mathfrak{X}(\mathbb{R}^n)^* & & \mathfrak{X}(S)^*
\end{tikzcd}$$

We also note that this framework shows that the parameterization of the singular solutions in terms of $\mathbf{Q}$ and $\mathbf{P}$ are exactly Clebsch variables in the sense given in Marsden and Weinstein [1983]. Also notice that when we write the singular solutions in $\mathbf{Q}$-$\mathbf{P}$ space, we are finding solutions that are *collective* and so all the properties of collectivization are valid. See Marsden and Ratiu [1999] for a general discussion and references to the original work of Guillemin and Sternberg on this topic.

6 Challenges, Future Directions and Speculations

Numerical Issues: Geometric Integrators. The computations of Martin Staley that were illustrated in this paper and that are discussed in Holm and Staley [2003], make use of both mimetic differencing and reversibility in a critical way and this is important for good numerical resolution. In other words, integrators that respect the basic geometry underlying the problem seem to play an key role. It would be interesting to pursue this aspect further and also incorporate discrete exterior calculus and variational multisymplectic integration methods (see Desbrun, Hirani, Leok and Marsden [2003] as well as Marsden, Patrick and Shkoller [1998] and Lew, Marsden, Ortiz and West [2003]).

Analytical Issues: Geodesic Incompleteness of H^1 EPDiff. The emergence in finite time of singular solutions from smooth initial data observed numerically in Holm and Staley [2003] indicates that the diffeomorphism group with respect to the right invariant H^1 metric is *geodesically incomplete* when the diffeomorphism group has the H^s topology, $s > (n/2) + 1$. The degree of its geodesic incompleteness is not known, but we suspect that almost all EPDiff geodesics in H^1 cannot be extended indefinitely. This certainly holds in one spatial dimension, where the discreteness of the CH isospectrum implies that asymptotically in time the CH solution arising from any confined initial velocity data consists *only* of peakons. *It is an important challange to find a context in which one can put the H^1 topology on the diffeomorphism group and reestablish geodesic completeness.* The numerics suggests that this might be possible, while known existence theorems, even for the CH equation are not yet capable of showing this—to the best of our knowledge.

Reversible Reconnections of the Singular EPDiff Solutions. EPDiff is a reversible equation, and the collisions of its peakon solutions on the line $\mathbb{R}^1$ (or the circle S^1) are known to be reversible. In principle, the reconnections of the singular EPDiff solutions observed numerically in Holm and Staley [2003] in periodic domains $\mathbb{T}^2$ and $\mathbb{T}^3$ must also be reversible. Reversibility of its reconnections distinguishes the singular solutions of EPDiff from vortex fluid solutions and shocks in fluids, whose reconnections apparently require dissipation and so, are not reversible. The mimetic finite differencing scheme used for the numerical computation of EPDiff solutions in Holm and Staley [2003] was indeed found to be reversible for overtaking collisions, but it was found to be only approximately reversible for head-on collisions, which are much more challenging for numerical integration schemes.

Applications of EPDiff Singular Solutions in Image Processing. The singular EPDiff solutions correspond to outlines (or cartoons) of images in applications of geodesic flow for the template, or pattern matching approach. The dynamics of the singular EPDiff solutions described by the momentum map (2.8) introduces the paradigm of soliton collisions into the mechanics and analysis of image processing by template matching. (See Holm, Trouvé and Younes [2003] for more discussions of this new paradigm for image processing.) The reversibility of the collisions among singular solutions and their reconnections under EPDiff flow assures the preservation of the information contained in the image outlines. In addition, the invariance of the manifold of N singular solutions under EPDiff assures that the fidelity of the image is preserved in the sense of approximation theory. That is, an N soliton approximation of the image outlines remains so, throughout the EPDiff flow. A natural approach for numerically simulating EPDiff flows in image processing is to use multisymplectic algorithms. The preservation of the space-time multisymplectic form by these algorithms introduces an initial-value, final-value formulation of the numerical solution procedure that is natural for template matching.

Rigorous Poisson Structures. In Vasylkevych and Marsden [2003], the question of the (rigorous) Poisson nature of the time t map of the flow of the Euler equations for an ideal fluid in appropriate Sobolev spaces is explored. Given the smoothness properties in §3, it seems reasonable that similar properties should also hold for the EPDiff equations. However, as mentioned earlier, these smoothness properties do not preclude the emergence of singular solutions from smooth initial data in finite time, because of the possibility for geodesic incompleteness.

Other Groups. The general setting of this paper suggests that perhaps one should look for similar measure valued or singular solutions associated with other problems, including geodesic flows on the group of symplectic diffeomorphisms (relevant for plasma physics, as in Marsden and Weinstein [1982]), Bott-Virasoro central extensions and super-symmetry groups.

Scattering. It might be interesting to explore the relation of the singular solution momentum map (2.8) to integrability and scattering data. For example, see

Vaninsky [2003] for an interesting discussion of the Poisson bracket for the scattering data of CH in 1D. This turns out to be the Atiyah-Hitchin bracket, which is also related to the Toda lattice, and this fascinating observation leads to an infinite-dimensional version of Jacobi elliptic coordinates.

Other Issues. Of course there are many other issues remaining to explore that are suggested by the above setting, such as convexity of the momentum map, its extension to Riemannian manifolds, etc. We shall, however, leave these issues for other publications and other researchers.

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